# Chapter 1

# Additive functionals and push forward measures under Veretennikov's flow

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*Dedicated to Professor Masatoshi Fukushima with admiration*

In this work, we will be interested in the push forward measure $(\varphi_t)_*\gamma$, where $\varphi_t$ is defined by the stochastic differential equation

$$d\varphi_t(x) = dW_t + \mathbf{a}(\varphi_t(x))dt, \quad \varphi_0(x) = x \in \mathbb{R}^m,$$

and $\gamma$ is the standard Gaussian measure. We will prove the existence of density under the hypothesis that the divergence $\mathrm{div}(\mathbf{a})$ is not a function, but a signed measure belonging to a Kato class; the density will be expressed with help of the additive functional associated to $\mathrm{div}(\mathbf{a})$.

## 1.1. Introduction

Let $(X_t)_{t\geq 0}$ be a Brownian flow on $\mathbb{R}^m$, that is, $W_t = X_t - X_0$ is a standard Brownian motion; then for a function $u \in C^2(\mathbb{R}^m)$, Itô formula says that

$$u(X_t) - u(X_0) = \int_0^t \nabla u(X_s)\cdot dW_s + \int_0^t \frac{1}{2}\Delta u(X_s)\, ds. \tag{1.1}$$

In a celebrated paper[13] M. Fukushima extended a $C^2$ function $u$ in (1.1) to a function $u$ in the Sobolev space $H^1(\mathbb{R}^m)$; in order to reach this end, he used an additive functional $N_t^{[u]}$ of $X.$ to express the last term in (1.1), moreover he showed that $N_t^{[u]}/t$ tends to $\frac{1}{2}\Delta u$ in distribution sense.

In this work, we will be concerned with the stochastic differential equation (SDE) on $\mathbb{R}^m$

$$d\varphi_t(x) = dW_t + \mathbf{a}(\varphi_t(x))dt, \quad \varphi_0(x) = x \in \mathbb{R}^m, \tag{1.2}$$

where $\mathbf{a} : \mathbb{R}^m \to \mathbb{R}^m$ is a measurable map.

The SDE (1.2), due to the non-degenerated noise $W_t$, makes illuminating difference with ordinary differential equations (ODE). In the context of ODE, the existence of a flow of quasi-invariant measurable maps associated to a vector field $\mathbf{a}$ on $\mathbb{R}^m$ belonging to Sobolev space, having a bounded divergence div($\mathbf{a}$), was established in a seminal paper by Di Perna and Lions in Ref.[8] ; their result was extended later in Ref.[1] by L. Ambrosio to a vector field having only bounded variation regularity and bounded divergence (see also Ref.[7]).

There are various considerations to SDE (1.2). When $\mathbf{a}$ is bounded, it was proved by Veretennikov in Ref.[25] that there exists a unique strong solution $\varphi_t(x)$ to SDE (1.2). Moreover if $\mathbf{a}$ is Hölderian, it was proved in Ref.[12] as well as in Ref.[27] that $x \to \varphi_t(x)$ is a flow of diffeomorphisms. Recently, it was proved in Ref.[3] that if $\mathbf{a}$ is of bounded variation, and $\mu_{k,j} = \frac{\partial a_k}{\partial x_j}$ are signed measures satisfying (1.19) for all $k, j$,, then the solution $\varphi_t$ to SDE (1.2) is in Sobolev space:

$$\varphi_t(\cdot) \in \cap_{p\geq 1} W^1_{p,loc}(\mathbb{R}^m, \mathbb{R}^m), t \geq 0.$$

Moreover, the Sobolev derivative $\nabla\varphi_t$ is a solution to the equation

$$\nabla\varphi_t = I + \int_0^t \bar{A}^\varphi(ds)\nabla\varphi_s(x), t \geq 0,$$

where $\bar{A}^\varphi$ is the additive functional associated to $\nabla\mathbf{a}$.

In Ref.[26] , X. Zhang allowed $\mathbf{a}$ to be time-dependent, and established the existence of strong solutions under integrability conditions on the drift $\mathbf{a}$, while in Ref,[19] Krylov and Röckner considered such a SDE on a domain of $\mathbb{R}^m$ and established the existence of strong solutions. In another direction, in Ref.[4] Bass and Chen took the point of view of additive functionals

$$A^i_t = \int_0^t \mathbf{a}^i(\varphi_s(x))\, ds,$$

where $\mathbf{a}^i$ denotes the $ith$-component of $\mathbf{a}$, to generalize the drift $\mathbf{a}$; $\mathbf{a}^i(x)dx$ seen as the Revuz measure associated to $A^i_t$, was extended to the Kato class $K_\alpha$ for some $\alpha > 0$, where $K_\alpha$ is the class of signed measures on $\mathbb{R}^m$ defined by

$$K_\alpha = \Big\{\pi(dx);\ \overline{\lim_{\varepsilon\to 0}} \sup_{x\in\mathbb{R}^m} \int_{B(x,\varepsilon)} |x-y|^\alpha\ |\pi|(dy) = 0\Big\} \tag{1.3}$$

where $|\pi|$ denotes the total variation of $\pi$. More precisely, they proved that if the Revuz measures $\pi^1, \ldots, \pi^m$ are in $K_{m-1}$ with $m \geq 3$, then with help of associated additive functionals $A_t = (A^1_t, \ldots, A^m_t)$, the SDE

$$X_t = x + W_t + A_t$$

admits a unique weak solution. The interest of considering $\pi^1, \ldots, \pi^m$ in Kato class is they are not necessarily absolutely continuous with respect to the Lebesgue measures. In the case where $\mathbf{a} = \nabla \log \rho$, by considering

$$\mathcal{E}(u,v) = \int_{\mathbb{R}^m} \nabla u \cdot \nabla v \ \rho(x)dx,$$

the theory of Dirichlet forms provides a powerful tool, which allows $\rho$ to be only locally integrable (see Refs.[14,15]).

In this work, we are interested in push-forward measures under the map $x \to \varphi_t(x)$ defined by SDE (1.2). It is well-known that if $\mathbf{a}$ is smooth and bounded, then $x \to \varphi_t(x)$ is a diffeomorphism of $\mathbb{R}^m$ and the inverse flow $\varphi_t^{-1}$ can be expressed by SDE with reversed Brownian motion. More precisely, for $t > 0$ given, let $W_s^t = W(t-s) - W(t)$, and $\psi_s^t$ solve the SDE

$$d\psi_s^t(x) = -\mathbf{a}(\varphi_s^t(x))dt + dW_s^t,\ \ s \in [0,t],\ \ \psi_0^t(x) = x; \tag{1.4}$$

then $\varphi_t^{-1} = \psi_t^t$. Let $\gamma$ be the standard Gaussian measure on $\mathbb{R}^m$. By Kunita[21] the push forward measure $(\varphi_t^{-1})_*\gamma$ admits the density $\tilde{K}_t$ with respect to $\gamma$ given by

$$\tilde{K}_t(x) = \exp\Big(-\int_0^t \langle \varphi_s(x), \circ dW_s\rangle - \int_0^t \delta(\mathbf{a})(\varphi_s(x))\, ds\Big) \tag{1.5}$$

where $\circ dW_s$ means the stochastic integral in Stratanovich's sense, and $\delta(\mathbf{a})$ is the divergence with respect to $\gamma$, that is,

$$\int_{\mathbb{R}^m} \langle \nabla f, \mathbf{a}\rangle\, d\gamma = \int_{\mathbb{R}} f\delta(\mathbf{a})\, d\gamma \quad \text{for all } f \in C_0^1(\mathbb{R}^m).$$

We have $\delta(\mathbf{a})(x) = \langle \mathbf{a}, x\rangle - \mathrm{div}(\mathbf{a})$ so that

$$\int_0^t \delta(\mathbf{a})(\varphi_s(x))\, ds = \int_0^t \langle \mathbf{a}(\varphi_s(x)), \varphi_s(x)\rangle\, ds - \int_0^t \mathrm{div}(\mathbf{a})(\varphi_s(x))\, ds. \tag{1.6}$$

Here is the main result of this paper

**Theorem 1.1.** *Let* $\mathbf{a} : \mathbb{R}^m \to \mathbb{R}^m$ *be a bounded measurable map. Assume that the divergence* $\mathrm{div}(\mathbf{a})$ *in generalized sense is a signed measure* $\mu$ *satisfying the condition*

$$\lim_{t\to 0} \sup_{x\in\mathbb{R}^m} \int_{\mathbb{R}^m} \Big(\int_0^t s^{-m/2} e^{-|x-y|^2/(2s)}\, ds\Big)\, |\mu|(dy) = 0, \tag{1.7}$$

*where* $|\mu|$ *denotes the total variation of* $\mu$. *Let* $\varphi_t$ *be given by SDE* (1.2)*; then almost surely the push forward measure* $(\varphi_t)_*\gamma$ *is equivalent to* $\gamma$, *and the density* $\tilde{K}_t$ *of the push forward measure* $(\varphi_t^{-1})_*\gamma$ *with respect to* $\gamma$ *has the expression*

$$\tilde{K}_t(x) = \exp\Big(-A_t + \int_0^t \langle \mathbf{a}(\varphi_s(x)), \varphi_s(x)\rangle\, ds - \int_0^t \langle \varphi_s(x), dW_s\rangle - \frac{mt}{2}\Big), \tag{1.8}$$

*where* $A_t$ *is the additive functional associated to* $\mathrm{div}(\mathbf{a})$.

Notice that if $f$ is a positive function in the Kato class $K_{m-2}$, then (see Ref.[4]),

$$\lim_{t\to 0}\sup_{x\in\mathbb{R}^m}\mathbb{E}\Big(\int_0^t f(W_s+x)\,ds\Big)=0,$$

that is nothing but (1.7) for $\mu(dy)=f(y)dy$.

The organization of the paper is as follows. In section 1.2, we will recall and collect some facts concerning continuous additive functionals. Section 1.3 is devoted to the proof of Theorem 1.1. In section 1.4, we will consider some examples of vector fields $\mathbf{a}$ whose divergence $\mathrm{div}(\mathbf{a})$ is a signed measure, but is not absolutely to Lebesgue measure. In section 1.5, we will discuss briefly generalizations of Theorem 1.1.

## 1.2. Continuous additive functionals

In this section, we recall some definitions and facts about continuous additive functionals of Markov processes. There are a lot of publications in the litterature on this topic, see for example Refs.[9,14,16,17,23,24] . Here we will follow Chapters 6 - 8 in Ref.[9] Chapter II, section 6 in Ref.[17] . We don't need the theory on the whole generality; so some assumptions, statements or definitions are simplified in our exposition.

Let $\{X_t, t\geq 0\}$ be a continuous $\mathbb{R}^m$-valued homogeneous Markov processes adapted to a filtration $\{\mathcal{F}_t, t\geq 0\}$ with infinite life-time, $P_x$ be the distribution of $X_\cdot$ given $X_0=x$. Denote $\mathcal{N}_t=\sigma(X_s, s\in[0,t])$.

**Definition 1.1.** A non negative additive functional of $X$ is a $\mathbb{R}_+$-valued, $\mathcal{N}_t$-adapted process $A=\{A_t(X), t\geq 0\}$ such that

1) it is almost surely continuous in $t$ and $A_0(X)=0$;
2) it is additive, i.e. $\forall t\geq 0\ \forall s\geq 0\ \forall x\in\mathbb{R}^m$ :

$$A_{t+s}(X)=A_s(X)+A_t(\theta_s X),\quad P_x\text{-a.s.},$$

where $\theta_s$ is the shift operator.

Following the terminology of Dynkin[9] , we introduce the notion of $W$-functional.

**Definition 1.2.** A non negative continuous additive functional $A_t(X)$ is called $W$-functional if

$$\forall t\geq 0:\ \sup_x \mathbb{E}_x(A_t(X))<\infty. \tag{1.9}$$

The function $f_t(x)=\mathbb{E}_x(A_t(X))$ is called the characteristics of $A_t(X)$.

Here is an obvious example

**Example 1.1.** Let $b:\mathbb{R}^m\to[0,\infty)$ be a bounded measurable function. Then

$$A_t(X):=\int_0^t b(X_s)ds \tag{1.10}$$

is a $W$-functional of $X$.

Assume that for any $t>0$, $X_t$ has a transition density $p(t,x,y)$. Then the characteristics of $A_t(X)$ defined in (1.10) is equal to

$$\begin{aligned} f_t(x) = \mathbb{E}_x \int_0^t b(X_s)ds = \int_0^t \int_{\mathbb{R}^m} b(y)p(s,x,y)dyds \\ = \int_{\mathbb{R}^m} \Big( \int_0^t p(s,x,y)ds \Big) b(y)dy. \end{aligned} \tag{1.11}$$

There are a close relations between convergence of $W$-functionals and their characteristics. The first one is the following

**Proposition 1.1.** (see Ref.[9] , Theorem 6.3) *A $W$-functional is defined by its characteristics uniquely up to the equivalence.*

The second one concerns the convergence, that is,

**Theorem 1.2.** (Ref.[9] , Theorem 6.4, Lemma 6.1′) *Let $\{A_t^{(n)}(X)\}$ be a sequence of $W$-functionals of $X$ and $f_t^{(n)}(x) = \mathbb{E}_x\big(A_t^{(n)}(X)\big)$ be their characteristics. Assume that a function $f_t(x)$ is such that for each $t>0$*

$$\lim_{n\to\infty} \sup_{0\le s\le t} \sup_{x\in\mathbb{R}^m} |f_s^{(n)}(x) - f_s(x)| = 0. \tag{1.12}$$

*Then $f_t(x)$ is the characteristics of a $W$-functional $A_t(X)$. Moreover, for each $t>0$,*

$$\lim_{n\to+\infty} \mathbb{E}_x(|A_t(X) - A_t^{(n)}(X)|^2) = 0,$$

*and in probability,*

$$\lim_{n\to+\infty} \sup_{s\in[0,t]} |A_s^{(n)}(X) - A_s(X)| = 0.$$

**Example 1.2.** Let $\{X_t = B_t,\, t\ge 0\}$ be a one-dimensional Brownian motion; set

$$A_t^{(n)} := \int_0^t 2n\mathbf{1}_{B(s)\in\left[-\frac{1}{n},\frac{1}{n}\right]} ds.$$

Then a function $b_n$ in expression (1.10) is equal to $2n\mathbf{1}_{\{|x|\le 1/n\}}$ and converges to the Dirac mass $\delta_0$ at 0. It is easy to verify that (1.12) holds with

$$f_t(x) = \int_{\mathbb{R}} \int_0^t p(s,x,y)ds\delta_0(dy) = \int_0^t p(s,x,0)ds,$$

where

$$p(s,x,y) = \frac{1}{\sqrt{2\pi s}} \exp\Big\{ \frac{|x-y|^2}{2s} \Big\}$$

is the transition density of a Brownian motion. The limiting additive functional is the local time of a Brownian motion at 0. Now let us write (1.11) as

$$f_t(x) = \int_{\mathbb{R}^d} \Big( \int_0^t p(s,x,y)ds \Big) \mu(dy), \tag{1.13}$$

with $\mu(dy) = a(y)dy$. Note that representation (1.13) makes a sense even $\mu$ is not absolutely continuous with respect to Lebesgue measure.

Similarly to Example 1.2, sometimes it is possible to assign a $W$-functional to a measure. For example, assume that there exists a sequence of non-negative bounded, measurable functions $\{b_n, n \geq 1\}$ such that for $t > 0$

$$\lim_{n\to 0} \sup_x \Big| \int_{\mathbb{R}^m} \left( \int_0^t p(s,x,y)ds \right) (\mu_n(dy) - \mu(dy)) \Big| = 0,$$

where $\mu_n(dy) = b_n(y)dy$. Then a function $f_t(x)$ defined in (1.13) is the characteristic of a $W$-functional. We will formally denote it by

$$A_t := \int_0^t \frac{d\mu}{dy}(X_s)ds.$$

If there are some a priori estimates on the transition density of $X_t$, then using the described approach it is possible to characterize a class of measures corresponding to its $W$-functionals. See for example Ref.[9] , Ch.8 for $W$-functionals of a Brownian motion.

Let's come back to SDE (1.2). It is known in Ref.[2] that the transition density of $\varphi_t(x)$ exists and there are constants $c_1, c_2 > 0$ depending only on $\sup_x |\mathbf{a}(x)|$ such that $\forall t \in (0,T]$,

$$c_1^{-1} t^{-m/2} \exp\Big\{ -\frac{|x-y|^2}{c_2 t} \Big\} \leq p(t,x,y) \leq c_1 t^{-m/2} \exp\Big\{ -\frac{c_2|x-y|^2}{t} \Big\}. \quad (1.14)$$

Observe that ( Ref.[9] Ch.8)

$$\int_0^t p(s,x,y)ds \asymp \omega(|x-y|),$$

where

$$\omega(r) = \begin{cases} 1, m = 1, \\ \ln(r-1) \vee 1, m = 2, \\ r^{2-m}, m \geq 3, \end{cases}$$

More precisely, for each $t > 0$, there exists a positive constant $C$ such that for all $x \neq y \in \mathbb{R}^m$ with $m > 1$,

$$C^{-1}\omega(|x-y|) \leq \int_0^t p(s,x,y)ds \leq C\omega(|x-y|). \quad (1.15)$$

So, a function $f_t(x)$ defined in (1.13) is finite if and only if $\int_{\mathbb{R}^m} \omega(x-y)\mu(dy) < \infty$. Hence, assumption (1.9) is equivalent to

$$\sup_x \int_{\mathbb{R}^m} \omega(|x-y|)\mu(dy) < \infty. \quad (1.16)$$

Assume that (1.16) is satisfied. It follows from ( Ref.[9] , Theorem 6.6) that condition

$$\lim_{t\to 0}\sup_x f_t(x) = 0 \tag{1.17}$$

ensures that $f_t(x) = \int_{\mathbb{R}^m}\Big(\int_0^t p(s,x,y)ds\Big)\mu(dy)$ is a characteristic of $W$-functional. It follows from (1.14) that (1.17) is equivalent to

$$\lim_{t\to 0}\sup_x \int_{\mathbb{R}^m}\Big(\int_0^t s^{-m/2}\exp\Big\{-\frac{|x-y|}{2s}\Big\}ds\Big)\mu(dy) = 0. \tag{1.18}$$

**Remark 1.1.** If $\mu$ satisfies (1.18), then $\mu$ satisfies (1.16).

Now we deal with signed additive functionals.

**Definition 1.3.** We say that $A_t(X)$ is a signed continuous additive functional if it has the decomposition $A_t(X) = A_t^+(X) - A_t^-(X)$, where $\{A_t^{\pm}(X), t\ge 0\}$ are continuous non negative additive functionals of $X$.

For a signed measure $\mu = \mu^+ - \mu^-$ such that

$$\lim_{t\to 0}\sup_x \int_{\mathbb{R}^m}\Big(\int_0^t s^{-m/2}\exp\Big\{-\frac{|x-y|^2}{2s}\Big\}ds\Big)|\mu|(dy) = 0, \tag{1.19}$$

where $|\mu|$ is the total variation of $\mu$, we can construct a signed W-functional $A_t = A_t^+ - A_t^-$, where functionals $A_t^+, A_t^-$ correspond to $\mu^+, \mu^-$ respectively (see Ref.[9]).

## 1.3. Proof of Theorem 1.1

Let $\mathbf{a}$ be a bounded measurable vector field on $\mathbb{R}^m$.

**Definition 1.4.** We say that a signed measure $\mu$ on $\mathbb{R}^m$ is the divergence in a generalized sense of $\mathbf{a}$ if for any test function $g \in C_0^\infty(\mathbb{R}^m)$ :

$$\int_{\mathbb{R}^m}\langle \mathbf{a}(x), \nabla g(x)\rangle dx = -\int_{\mathbb{R}^m} g(x)\mu(dx),$$

where $dx$ on the left hand side denotes the Lebesgue measure; we denote $\mu = \mathrm{div}(\mathbf{a})$.

In what follows, we will assume that $\mathrm{div}(\mathbf{a})$ exists and satisfies condition (1.7).

Let $\{g_n, n\ge 1\} \subset C_0^\infty(\mathbb{R}^m)$ be a sequence of non-negative smooth functions with compact support such that

$$\int_{\mathbb{R}^m} g_n(x)dx = 1, \text{ and } g_n(x) = 0 \text{ for } |x| > \frac{1}{n}.$$

Put

$$\mathbf{a}_n(x) := \mathbf{a} * g_n(x) = \int_{\mathbb{R}^m}\mathbf{a}(x-y)g_n(y)dy. \tag{1.20}$$

Note that $\mathbf{a}_n \in C^\infty(\mathbb{R}^m, \mathbb{R}^m)$,

$$\|\mathbf{a}_n\|_\infty = \sup_x |\mathbf{a}_n(x)| \le \sup_x |\mathbf{a}(x)| = \|\mathbf{a}\|_\infty, \tag{1.21}$$

and $\mathbf{a}_n$ converges to $\mathbf{a}$ in all $L^p_{loc}(\mathbb{R}^m, \mathbb{R}^m)$. Without loss of generality we may assume that as $n \to +\infty$,

$$\mathbf{a}_n(x) \to \mathbf{a}(x) \text{ for almost everywhere } x \in \mathbb{R}^m. \tag{1.22}$$

Let $\varphi_t^n(x)$ be the stochastic flow of diffeomorphisms defined by

$$d\varphi_t^n(x) = dW_t + \mathbf{a}_n(\varphi_t^n(x))dt, \quad \varphi_0^n(x) = x \in \mathbb{R}^m. \tag{1.23}$$

Let $\gamma$ be the standard Gaussian measure on $\mathbb{R}^m$. We set

$$K_t^n(x) = \frac{d(\varphi_t^n)_*\gamma}{d\gamma}, \quad \tilde{K}_t^n(x) = \frac{d(\varphi_t^n)_*^{-1}\gamma}{d\gamma}.$$

It is well-known (see Ref.[21]) that

$$K_t^n(\varphi_t^n(x)) = \frac{1}{\tilde{K}_t^n(x)}, \tag{1.24}$$

and

$$\begin{aligned}\tilde{K}_t^n(x) &= \exp\Big\{ -\int_0^t (\delta\mathbf{a}_n)(\varphi_s^n(x))ds - \int_0^t \langle \varphi_s^n(x), \circ dW_s\rangle \Big\} \\ &= \exp\Big\{ -\int_0^t (\delta\mathbf{a}_n)(\varphi_s^n(x))ds - \int_0^t \langle \varphi_s^n(x), dW_s\rangle - \frac{mt}{2}\Big\},\end{aligned} \tag{1.25}$$

where $\delta\mathbf{a}_n(x) = (\operatorname{div}\mathbf{a}_n)(x) - \langle \mathbf{a}_n(x), x\rangle$. In,[10] the $L^p$ estimates on densities were established and used to prove the absolute continuity for a limit of push-forward measures. Here we will use the following result of Gikhman and Skorokhod Ref.[18] .

**Theorem 1.3.** *(see* Ref.[18]*) Let $(X_1.\mathcal{F}, \mu_1)$ be a probability space, $X_2$ be a complete separable metric space, $\mu_2$ be a probability measure on the Borel $\sigma$-algebra $\mathcal{B}(X_2)$. Assume that a sequence of measurable mappings $\{F_n : X_1 \to X_2, n \ge 0\}$ is such that*

*1) as $n \to +\infty$, $F_n$ converges to $F_0$ in measure $\mu_1$;*

*2) for all $n \ge 1$, the push forward measure $(F_n)_*\mu_1$ is absolutely continuous with respect to $\mu_2$;*

*3) the sequence of the densities $\Big\{\rho_n := \frac{d(F_n)_*\mu_1}{d\mu_2}, n \ge 1\Big\}$ is uniformly integrable with respect to $\mu_2$.*

*Then the push forward measure $(F_0)_*\mu_1$ is absolutely continuous with respect to $\mu_2$. Moreover, if $\rho_n$ converges to $\rho$ in measure $\mu_2$, then $\rho = \frac{d(F_0)_*\mu_1}{d\mu_2}$.*

Let us apply Theorem 1.3 to the sequence $\{\varphi_t^n, n \geq 1\}$. First of all, we remark that although in Ref.[22] D. Luo assumed that the drift admits the divergence as a function satisfying some integrability condition, but in the proof of Theorem 3.4 in Ref.,[22] he only used Krylov estimate for non-degenerated diffusions, without involving the divergence. Since $\mathbf{a}_n$ converges to $\mathbf{a}$ in all $L^p_{loc}$, we can use directly Theorem 3.4 in Ref.[22] to get that for each $x \in \mathbb{R}^m$, we have

$$\lim_{n\to+\infty} \mathbb{E}\Big(\sup_{t\in[0,T]} |\varphi_t^n(x) - \varphi_t(x)|\Big) = 0. \tag{1.26}$$

Applying Fubini's theorem and choosing a subsequence if necessary we get

$$P\Big(\Big\{\omega;\ \text{for } \gamma \text{ almost surely } \ x, \lim_{n\to\infty} \sup_{t\in[0,T]} |\varphi_t^n(x) - \varphi_t(x)| = 0\Big\}\Big) = 1. \tag{1.27}$$

It follows that for almost surely $\omega$, for all $t \in [0,T]$, $\varphi_t^n$ converges to $\varphi_t$ in measure with respect to $\gamma$.

Next, we will establish the uniform integrability of $\{K_t^n;\ n \geq 1\}$.

**Proposition 1.2.** *We have*

$$\sup_{n\geq 1} \mathbb{E}\Big(\sup_{t\in[0,T]} \int_{\mathbb{R}^m} K_t^n(x) |\ln K_t^n(x)| \gamma(dx)\Big) < +\infty. \tag{1.28}$$

**Proof.** We have

$$\int_{\mathbb{R}^m} K_t^n(x) |\ln(K_t^n(x))|\, d\gamma(x) = \int_{\mathbb{R}^m} |\ln(K_t^n(\varphi_t^n(x)))|\, d\gamma(x).$$

But by (1.24) and (1.25), we have

$$\begin{aligned}\ln(K_t^n(\varphi_t^n(x))) = \int_0^t (\mathrm{div}(\mathbf{a}_n)(\varphi_s^n(x))\, ds - \int_0^t \langle \mathbf{a}_n(\varphi_s^n(x)), \varphi_s^n(x)\rangle\, ds \\ + \int_0^t \langle \varphi_s^n(x), dW_s\rangle + \frac{mt}{2}.\end{aligned}$$

Let $T > 0$ be fixed; then

$$\begin{aligned}&\mathbb{E}\Big(\sup_{t\in[0,T]} \int_{\mathbb{R}^m} K_t^n(x) |\ln K_t^n(x)| \gamma(dx)\Big) \\ &\leq \int_{\mathbb{R}^m} \Big[\mathbb{E}\big(\sup_{t\in[0,T]} |\int_0^t (\mathrm{div}\, \mathbf{a}_n)(\varphi_s^n(x)) ds|\big) + \mathbb{E}\big(\sup_{t\in[0,T]} |\int_0^t \langle \varphi_s^n(x), dW_s\rangle|\big) \\ &+ \mathbb{E}\big(\int_0^T |\mathbf{a}_n(\varphi_s^n(x))|\, |\varphi_s^n(x)| ds\big) + mT\Big]\, d\gamma(x).\end{aligned} \tag{1.29}$$

By (1.21), it is well-known that there exists a constant $c_0 > 0$ independent of $n$ such that

$$\sup_n \mathbb{E}\Big(\sup_{t\in[0,T]} |\varphi_t^n(x)|\Big) \leq c_0(1+|x|), \tag{1.30}$$

Using Burkholder's inequality and (1.30), we also have

$$\sup_n \mathbb{E}\Big(\sup_{t\in[0,T]} |\int_0^t \langle \varphi_s^n(x), dW_s\rangle|\Big) \leq c_0(1+|x|). \tag{1.31}$$

Let us estimate $\mathbb{E}\Big(\sup_{t\in[0,T]} |\int_0^t (\operatorname{div} \mathbf{a}_n)(\varphi_s^n(x))ds|\Big)$.
Denote

$$\mu(dy) = (\operatorname{div} \mathbf{a})(dy),\ \mu_n(dy) = \operatorname{div} \mathbf{a}_n(y)dy.$$

We have

$$\operatorname{div}(\mathbf{a}_n) = \operatorname{div}(\mathbf{a} * g_n) = \operatorname{div} \mathbf{a} * g_n = \mu * g_n.$$

Let

$$A_n(t) = \int_0^t (\operatorname{div} \mathbf{a}_n)(\varphi_s^n(x))ds;$$

then $A_n$ is a signed additive functional of $\varphi^n$. Let $p_n(t,x,y)$ be the transition density of $\omega \to \varphi_t^n(x,\omega)$. By (1.14), there are two constants $c_1, c_2 > 0$ independent of $n$ such that

$$\frac{e^{-|x-y|^2/c_2 2t}}{c_1 t^{m/2}} \leq p_n(t,x,y) \leq \frac{c_1 e^{-c_2|x-y|^2/2t}}{t^{m/2}}.$$

So there exists a constant $\beta > 0$ independent of $n$ such that

$$\int_0^t p_n(s,x,y)\, ds \leq \beta \int_0^{t/c_2} \frac{e^{-|x-y|^2/2s}}{s^{m/2}}\, ds.$$

Let

$$k_t(r) = \int_0^{t/c_2} \frac{e^{-r^2/2s}}{s^{m/2}}\, ds.$$

It follows that

$$\begin{aligned}\mathbb{E}(|A_n(t)|) &\leq \int_{\mathbb{R}^m}\int_0^t p_n(s,x,y)|\operatorname{div} \mathbf{a}_n(y)|\, dsdy \\ &\leq \beta \int_{\mathbb{R}^m} k_t(|x-y|)|\mu_n|(dy).\end{aligned} \tag{1.32}$$

We have

$$|\operatorname{div}(\mathbf{a}_n)(y)| \leq \int_{\mathbb{R}^m} g_n(y-z)\, |\mu|(dz),$$

so that

$$\int_{\mathbb{R}^m} k_t(|x-y|)|\mu_n(y)|\,dy \le \int_{\mathbb{R}^m\times\mathbb{R}^m} k_t(|x-y|)g_n(y-z)\,|\mu|(dz)dy$$
$$= \int_{\mathbb{R}^m}\Big(\int_{\mathbb{R}^m} k_t(|x+z-y|)g_n(y)\,|\mu|(dz)\Big)\,dy \le \sigma(t),$$

where

$$\sigma(t) = \sup_{x,y}\int_{\mathbb{R}^m} k_t(|x+z-y|)\,|\mu|(dz).$$

Then by condition (1.7), $C_T := \sup_{t\in[0,T]}\sigma(t) < +\infty$. Now combining this with (1.30) and (1.31), and by (1.29), we finally obtained (1.28). □

Now by Fatou's lemma,

$$\mathbb{E}\left(\varliminf_{n\to\infty}\sup_{t\in[0,T]}\int_{\mathbb{R}^m} K_t^n(x)|\ln K_t^n(x)|\nu(dx)\right)$$
$$\le \varliminf_{n\to\infty}\mathbb{E}\Big(\sup_{t\in[0,T]}\int_{\mathbb{R}^m} K_t^n(x)|\ln K_t^n(x)|\nu(dx)\Big) < \infty.$$

So, for almost surely $\omega$ :

$$\varliminf_{n\to\infty}\sup_{t\in[0,T]}\int_{\mathbb{R}^m} K_t^n(x)|\ln K_t^n(x)|\nu(dx) < \infty.$$

Hence for almost surely $\omega$, there exists a random subsequence $\{n_k\}$ such that

$$\sup_k\sup_{t\in[0,T]}\int_{\mathbb{R}^m} K_t^{n_k}(x)|\ln K_t^{n_k}(x)|\nu(dx) < \infty. \tag{1.33}$$

Now we can apply Theorem 1.3 to conclude that for almost surely $\omega$, and for all $t\in[0,T]$, the push-forward measure $(\varphi_t)_*\gamma$ is absolutely continuous with respect to $\gamma$. Actually it remains to prove that $(\varphi_t)_*\gamma$ is equivalent to $\gamma$.

**Proposition 1.3.** *The map $x\to\varphi_t(x)$ admits an inverse map $x\to\psi_t(x)$, which is given by the reserved SDE*

$$d\psi_s(x) = dW_s^t - \mathbf{a}(\psi_s(x))\,ds,\quad \psi_0(x)=x,\quad s\in[0,t]. \tag{1.34}$$

**Proof.** For each $n$, the inverse map of $x\to\varphi_t^n(x)$ is given by $\psi_t^n$ where $\psi_s^n$ solves

$$d\psi_s^n(x) = dW_s^t - \mathbf{a}(\psi_s^n(x))\,ds,\quad \psi_0^n(x)=x,\quad s\in[0,t].$$

In the same way, we have

$$\lim_{n\to+\infty}\mathbb{E}\Big(\sup_{s\in[0,t]}|\psi_s^n(x)-\psi_s(x)|\Big) = 0.$$

In order to prove that $\psi_t$ is the inverse map of $\varphi_t$, we will use the following result

**Lemma 1.1.** *Let $X, Y$ be complete, separable metric spaces, $\nu$ be a finite measure on $X$.*
*Assume that a sequence of $X$-valued random elements $\{\xi_n, n \geq 0\}$ and a sequence of measurable functions $f_n : X \to Y$ are such that*
*1) $\xi_n \to \xi_0$ in probability, as $n \to \infty$;*
*2) $f_n \to f_0$ in measure with respect to $\nu$, as $n \to \infty$;*
*3) the push forward measure $(\xi_n)_* P$ is absolutely continuous with respect to $\nu$;*
*4) the sequence of densities $\left\{ \frac{d(\xi_n)_* P}{d\nu}, n \geq 1 \right\}$ is uniformly integrable with respect to $\nu$.*
*Then as $n \to +\infty$*

$$f_n(\xi_n) \to f_0(\xi_0) \quad \textit{in probability.}$$

We refer to Corollary 9.9.11 in Ref.,[5] as well as to Lemma 2 in Ref.[20] for a proof. □

**Proof (continued) of Proposition 1.3.** For almost surely $\omega$, up to a subsequence, the family of densities $\{K_t^n;\ n \geq 1\}$ is uniformly integrable. In Lemma 1.1, we take $X = Y = \mathbb{R}^m$, $\xi_n = \varphi_t^n$, $f_n = \psi_t^n$. Then $\varphi_t^n(\psi_t^n)$ converges to $\varphi_t(\psi_t)$ in probability. So that $\varphi_t \circ \psi_t = Id$. In the same way, we prove that $\psi_t \circ \varphi_t = Id$. □

**End of the proof of Theorem 1.1.** Let $A_t$ be a signed additive functional of $\varphi_t$ that corresponds to the measure $\mu = \operatorname{div} \mathbf{a}$. Then using Theorem 1.2, similarly to the proof of Lemma 3 in Ref.[3] , we get

$$\int_0^t \operatorname{div} \mathbf{a}_n(\varphi_s^n(x)) ds \to A_t \text{ in } L^2, \text{ as } n \to \infty.$$

Using again Lemma 1.1, we have for $s$ fixed, $\mathbf{a}_n \circ \varphi_s$ converges to $\mathbf{a} \circ \varphi_s$ in measure. Therefore by expression

$$\begin{aligned}\tilde{K}_t^n(\omega, x) = \exp\Big\{ -\int_0^t (\operatorname{div} \mathbf{a}_n)(\varphi_s^n(x))\, ds + \int_0^t \langle \mathbf{a}_n(\varphi_s(x)), \varphi_s^n(x)\rangle\, ds \\ - \int_0^t \langle \varphi_s^n(x), dW_s\rangle - \frac{mt}{2}\Big\},\end{aligned}$$

when $n \to +\infty$, $K_t^n$ converges in measure $P \otimes \gamma$ to

$$\exp\Big\{ -A_t + \int_0^t \langle \mathbf{a}(\varphi_s(x)), \varphi_s(x)\rangle ds - \int_0^t \langle \varphi_s(x), dW_s\rangle - \frac{mt}{2}\Big\}. \tag{1.35}$$

The proof of Theorem 1.1 is completed. □

### 1.4. Examples

In this section, we will construct examples of vector fields $\mathbf{a}$ satisfying the condition in Theorem 1.1.

a) Examples of $W$-functionals.
Let $\{X(t), t \geq 0\}$ be a Markov process in $\mathbb{R}^m$ with transition density satisfying condition (1.14). Let $D_1, \dots, D_k$ be bounded domains of $\mathbb{R}^m$ with $C^1$ boundary, and $\sigma_{\partial D_j}$ be the surface measure on $\partial D_j$.
Let $\mu$ be a signed measure defined by

$$\mu(dx) = b_0(x)dx + \sum_{j=1}^{k} b_k(x)\sigma_{\partial D_j}(dx), \tag{1.36}$$

where $b_0, \dots, b_k$ are bounded measurable functions. Then conditions (1.16) and (1.18) are satisfied. So the additive functional

$$A(t) = \int_0^t \frac{d\mu(X(s))}{dx} ds$$

is well-defined.
Remark that for $m = 1$, any finite measure $\mu$ satisfies condition (1.18). Indeed,

$$\sup_x \int_{\mathbb{R}^m} \int_0^t s^{-1/2} \exp\Big\{ - \frac{|x-y|^2}{s} \Big\} ds\mu(dy) \leq \mu(\mathbb{R}) \int_0^t s^{-1/2} ds \to 0, t \to 0+.$$

b) Functions of bounded variation.
Assume that derivatives $\frac{\partial \mathbf{a}_i}{\partial x_k}$ considered in a generalized sense are measures. Such function $\mathbf{a} = (\mathbf{a}_1, \dots, \mathbf{a}_m)$ are called functions of bounded variation (BV). If this measures are of the form (1.36) with bounded $b_j$, then $\mathbf{a}$ satisfies condition (1.7).
Let now $g \in C^1(\mathbb{R}^m, \mathbb{R}^m), D$ be a bounded domain with $C^1$ boundary. Then $\mathbf{a}(x) = g(x)\mathbf{1}_{\{x\in D\}}$ also satisfies condition (1.7) since the generalized divergence $\operatorname{div} \mathbf{a}$ equals to

$$(\operatorname{div} g(x))\mathbf{1}_{\{x\in D\}}dx + \langle g(x), n(x)\rangle \sigma_{\partial D}(dx),$$

where $n(x)$ is the normal vector at $x \in \partial D$ (see Ref.[11]).
Linear combinations of the form

$$b_0(x) + \sum_{j=1}^{k} g_j(x)\mathbf{1}_{x\in D_j}$$

also satisfy condition (1.7), if $b_0 \in \text{Lip}, \ g_j \in C^1, D_j$ are bounded with $C^1$ boundary.

It should be noted that if $\mathbf{a} = (a_1, \dots, a_m)$ is a vector field of bounded variation and $\mu_{k,j} = \frac{\partial a_k}{\partial x_j}$ satisfies (1.19) for all $k, j$, it has been proved in Ref.[3]

$$P\big(\varphi_t(\cdot) \in \cap_{p\geq 1} W^1_{p,loc}(\mathbb{R}^m, \mathbb{R}^m), t \geq 0\big) = 1.$$

Moreover, the Sobolev derivative is a solution of the equation

$$\nabla\varphi_t = I + \int_0^t \bar{A}^{\varphi}(ds)\nabla\varphi_s(x), t \geq 0, \tag{1.37}$$

where

$$\bar{A}^{\varphi}(t) = \int_0^t \nabla \mathbf{a}(\varphi_s) ds, t \geq 0 \tag{1.38}$$

was defined in section 1.2. It follows from (1.37) that a.s.

$$\det \nabla \varphi_t(x) = \exp\{\mathrm{tr}(\bar{A}^{\varphi}(t))\} > 0.$$

Hence it follows from Ch. 9.2 in Ref.[6] the absolute continuity $(\varphi_t)_* \gamma$ with respect to $\gamma$.

c) Example of $\mathbf{a} \notin BV$ with $\operatorname{div} \mathbf{a} = 0$.
Functions of bounded variation is not unique example satisfying condition (1.7). For $m = 2$, let

$$\mathbf{a}(x_1, x_2) = (g(x_1 - x_2), g(x_2 - x_1)),$$

where $g$ is only measurable, bounded function. Then $\operatorname{div} \mathbf{a} = 0$ in the generalized sense, but partial derivatives $\frac{\partial \mathbf{a}}{\partial x_k}$ may not be measures.

### 1.5. Generalizations and localization.

In this section, we give briefly some generalization of Theorem 1.1.

Assume now that the vector field $\mathbf{a}$ is locally bounded and for any $x \in \mathbb{R}^m$. Assume that SDE (1.2) is conservative in the sense of Kunita[21] , that is, if $\tau(x)$ is the life-time of $\varphi_t(x)$, then

$$P(\{\omega; \tau(x) = +\infty\}) = 1.$$

For example, this is the case if $\mathbf{a}$ has a linear growth.

Let $\{f_n;\ n \geq 1\}$ be a sequence of functions in $C_0^\infty(\mathbb{R}^m)$ such that

$$\sup_{n,x}(|f_n(x)| + |\nabla f_n(x)|) < \infty;\ f_n(x) = 1, \text{ for } \ |x| \leq n.$$

Denote

$$\mathbf{a}_n(x) = \mathbf{a}(x) f_n(x),$$

and

$$\tau_n(x) = \inf\{t \geq 0 : |\varphi_t(x)| \geq n\}.$$

Let $\varphi_t^n$ be the solution to SDE (1.2) with $\mathbf{a}_n$ instead of $\mathbf{a}$. Observe that $\mathbf{a}_n$ is a bounded vector field on $\mathbb{R}^m$. By uniqueness of solutions, almost surely, for $t \leq \tau_n(x)$, $\varphi_t(x) = \varphi_t^n(x)$. So for any bounded Borel function $h : \mathbb{R}^m \to \mathbb{R}$,

$$\int_{\{\tau_n(x) \geq t\}} h(\varphi_t(x))\, d\gamma(x) = \int_{\{\tau_n(x) \geq t\}} h(\varphi_t^n(x))\, d\gamma(x). \tag{1.39}$$

Observe that $(\varphi_t^n)_*(\mathbf{1}_{\{\tau_n(x)\geq t\}}\gamma)$ is absolutely continuous with respect to $(\varphi_t^n)_*\gamma$ and for almost surely $\omega$, $(\varphi_t)_*(\mathbf{1}_{\{\tau_n(x)\geq t\}}\gamma)$ converges to $(\varphi_t)_*\gamma$ weakly as $n\to+\infty$ since $\tau_n(x)\to+\infty, n\to\infty$.

Assume that for each $n\geq 1$

$$\lim_{t\to 0}\sup_{|x|\leq n}\int_{|y|\leq n}\int_0^t s^{-m/2}\exp\Big\{-\frac{|x-y|^2}{2s}\Big\}ds\,|\mu|(dy)=0, \qquad (1.40)$$

where $\mu=\operatorname{div}\mathbf{a}$. Then for any $n$, the vector field $\mathbf{a}_n$ satisfies condition (1.7) in Theorem 1.1; therefore the push forward measure $(\varphi_t^n)_*\gamma$ is absolutely continuous with respect to $\gamma$. Now let $E$ be a Borel subset of $\mathbb{R}^m$ such that $\gamma(E)=0$; then by (1.39), then

$$\int_{\{\tau_n(x)\geq t\}}\mathbf{1}_E(\varphi_t(x))\,d\gamma(x)=\int_{\{\tau_n(x)\geq t\}}\mathbf{1}_E(\varphi_t^n(x))\,d\gamma(x)\leq[(\varphi_t^n)_*\gamma](E)=0.$$

Letting $n\to+\infty$ yields $[(\varphi_t)_*\gamma](E)=0$. in other words, $(\varphi_t)_*\gamma$ is absolutely continuous with respect to $\gamma$.
Note also that in this case

$$A_n^{\varphi^n}(t)=A_m^{\varphi^m}(t),\ t\in[0,\tau_n(x)]\text{ a.s.}$$

for all $m\geq n$. Therefore we can define $A^\varphi(t)=\lim_{n\to\infty}A_n^{\varphi^n}(t)$ and expression (1.8) also holds true if the reverse SDE (1.34) is conservative. □

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